LAST MULTIPLIERS AS AUTONOMOUS SOLUTIONS OF THE LIOUVILLE EQUATION OF TRANSPORT

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ABSTRACT. Using the characterization of last multipliers as solutions of the Liouville's transport equation, new results are given in this approach of ODEs by providing several new characterizations, e.g. in terms of Witten and Marsden differentials or adjoint vector field. Applications to Hamiltonian vector fields on Poisson manifolds and vector fields on Riemannian manifolds are presented. In the Poisson case, the unimodular bracket considerably simplifies computations while, in the Riemannian framework, a Helmholtz type decomposition yields remarkable examples: One is the quadratic porous medium equation, the second (the autonomous version of the previous) produces harmonic square functions, while the third refers to the gradient of the distance function with respect to a two dimensional rotationally symmetric metric. A final example relates the solutions of Helmholtz (particularly Laplace) equation to provide a last multiplier for a gradient vector field. A connection of our subject with gas dynamics in Riemannian setting is pointed out at the end.

INTRODUCTION

In January 1838, Joseph Liouville(1809-1882) published a note ([10]) on the time-dependence of the Jacobian of the "transformation" exerted by the solution of an ODE on its initial condition. In modern language, if $A = A(x)$ is the vector field corresponding to the given ODE and $m = m(t, x)$ is a smooth function

(depending also of the time t), then the main equation of the cited paper is:

$$\frac{dm}{dt} + m \cdot divA = 0 \qquad\qquad (LE)$$

called, by then, the *Liouville equation*. Some authors use the name *generalized Liouville equation* ([4]) but we prefer to name it the *Liouville equation of transport* (or *of continuity*). This equation is a main tool in statistical mechanics where a solution is called a *probability density function* ([22]).

The notion of *last multiplier*, introduced by Carl Gustav Jacob Jacobi (1804-1851) around 1844, was treated in detailed in "Vorlesungen über Dynamik", edited by R. F. A. Clebsch in Berlin in 1866. So that, sometimes it has been used under the name of "Jacobi multiplier". Since then, this tool for understanding ODE was intensively studied by mathematicians in the usual Euclidean space $I\!R^n$, cf. the bibliography of [3], [13]-[16]. For all those interested in historical aspects an excellent survey can be found in [2].

The aim of the present paper is to show that last multipliers are exactly the autonomous, i.e. time-independent, solutions of LE and to discuss some results of this useful theory extended to differentiable manifolds. Our study has been inspired by the results presented in [17] using the calculus on manifolds especially the Lie derivative. Since the Poisson and Riemannian geometries are the most frequently used frameworks, a Poisson bracket and a Riemannian metric are added and cases yielding last multipliers are characterized in terms of unimodular Poisson brackets and respectively, harmonic functions.

The paper is structured as follows. The first section reviews the definition of last multipliers and some previous results. New characterizations in terms of de Rham cohomology and other types of differentials than the usual exterior derivative, namely Witten and Marsden, are given. There follows that the last multipliers are exactly the first integrals of the adjoint vector field. For a fixed smooth function m, the set of vector fields admitting m as last multiplier is a Lie subalgebra of the Lie algebra of vector fields.

In the next section the Poisson framework is discussed, on showing that some simplifications are possible for the unimodular case. For example, in unimodular setting, the set of functions f which are last multiplier for exactly the Hamiltonian vector field generated by f is a Poisson subalgebra. Local expressions for the main results of this section are provided in terms of the bivector π defining the Poisson bracket. Two examples are given: one with respect to dimension two when as a last multiplier is obtained exactly the function defining the bivector, and the second related to Lie-Poisson brackets when the condition to be last multiplier is

expressed in terms of structural constants of given Lie algebra. Again, the two dimensional case is studied in detail and for non-vanishing structure constants, an affine function is obtained as a last multiplier.

The last section, is devoted to the Riemannian manifolds and again, some characterizations are given in terms of vanishing of some associated differential operators, e.g. the codifferential. Assuming a Helmholtz type decomposition, several examples are given: the first is related to a parabolic equation of porous medium type and the second yields harmonic square functions. As to the first example, one should notice the well-known relationship between the heat equation (in our case, a slight generalization) and the general method of multipliers; see the examples from [19, p. 364]. The third example is devoted to the distance function on a two dimensional rotationally symmetric Riemannian manifold, while the final example is concerned with solutions of Helmholtz (particularly Laplace) equation, for obtaining a last multiplier for a gradient vector field.

The last section deals with an application of our framework to gas dynamics on Riemannian manifolds but in order to not enlarge our paper too much we invite the reader to see details of physical nature in Sibner's paper [20]. More precisely, a main result from [20] is rephrased in terms of last multipliers.

1. General facts on last multipliers

Let M be a real, smooth, n-dimensional manifold, $C^\infty(M)$ the algebra of smooth real functions on M, $\mathcal{X}(M)$ the Lie algebra of vector fields and $\Lambda^k(M)$ the $C^\infty(M)$-module of k-differential forms, $0 \leq k \leq n$. Assume that M is orientable with the fixed volume form $V \in \Lambda^n(M)$.

Let:
$$\dot{x}^i(t) = A^i\left(x^1(t), \ldots, x^n(t)\right), 1 \leq i \leq n$$
an ODE system on M defined by the vector field $A \in \mathcal{X}(M)$, $A = \left(A^i\right)_{1 \leq i \leq n}$ and let us consider the $(n-1)$-form $\Omega = i_A V \in \Lambda^{n-1}(M)$.

Definition 1. ([6, p. 107], [17, p. 428]) The function $m \in C^\infty(M)$ is called a *last multiplier* of the ODE system generated by A, (*last multiplier* of A, for short) if $m\Omega$ is closed:
$$d(m\Omega) := (dm) \wedge \Omega + m\,d\Omega = 0. \tag{1.1}$$

For example, in dimension 2, the notions of last multiplier and integrating factor are identical and Sophus Lie suggests a method to associate a last multiplier to every symmetry vector field of A (Theorem 1.1 in [8, p. 752]). Lie method is extended to any dimension in [17].

If the $(n-1)^{th}$ de Rham cohomology space of M is zero, $H^{n-1}(M) = 0$, there follows that m is a last multiplier iff there exists $\alpha \in \Lambda^{n-2}(M)$ so that $m\Omega = d\alpha$. Other characterizations can be obtained in terms of Witten's differential [25] and Marsden's differential [11, p. 220]. If $f \in C^\infty(M)$ and $t \geq 0$, Witten deformation of the usual differential $d_{tf} : \Lambda^*(M) \to \Lambda^{*+1}(M)$ is defined by:

$$d_{tf} = e^{-tf} d e^{tf}$$

which means [25]:

$$d_{tf}(\omega) = tdf \wedge \omega + d\omega.$$

Hence, m is a last multiplier if and only if:

$$d_m \Omega = (1 - m) d\Omega$$

i.e. Ω belongs to the kernel of the differential operator $d_m + (m-1) d : \Lambda^{n-1}(M) \to \Lambda^n(M)$. Marsden differential is $d^f : \Lambda^*(M) \to \Lambda^{*+1}(M)$ defined by:

$$d^f(\omega) = \frac{1}{f} d(f\omega)$$

and m is a last multiplier iff Ω is d^m-closed.

The following characterization of last multipliers will be useful ([17, p. 428]):

Lemma 1.1. $m \in C^\infty(M)$ is a last multiplier for A if and only if:

$$A(m) + m \cdot div A = 0 \tag{1.2}$$

where $div A$ is the divergence of A with respect to volume form V. Let $0 \neq h \in C^\infty(M)$ such that:

$$L_A h := A(h) = (div A) \cdot h \tag{1.3}$$

Then $m = h^{-1}$ is a last multiplier for A.

Remark. (i) Equation (1.2) is exactly the time-independent version of LE from the Introduction. So that, the promised relationship between LE and last multipliers is obtained. An important feature of equation (1.2) is that it does not always admit solutions cf. [7, p. 269].

(ii) In the terminology of [2, p. 89], a function h satisfying (1.3) is called an *inverse multiplier*.

(iii) A first result given by (1.2) is the characterization of last multipliers for divergence-free vector fields: $m \in C^\infty(M)$ *is a last multiplier for the divergence-less vector field A iff m is a first integral of A*. The importance of this result is shown by the fact that three remarkable classes of divergence-free vector fields are provided by: Killing vector fields in Riemannian geometry, Hamiltonian vector fields in symplectic geometry and Reeb vector fields in contact geometry. Also, there are many equations of mathematical physics corresponding to the vector fields without divergence.

(iv) For the general case, namely A is not divergenceless, there is a strong connection between first integrals and last multipliers as well. Namely, from properties of Lie derivative, the ratio of two last multipliers is a first integral and conversely, the product between a first integral and a last multiplier is a last multiplier. So, denoting $FInt(A)$ the set of first integrals of A, since $FInt(A)$ is a subalgebra in $C^\infty(M)$ it results that the set of last multipliers for A is a $FInt(A)$-module.

(v) Recalling formula:

$$div\,(fX) = X\,(f) + f\,divX \qquad (1.4)$$

there follows that m is a last multiplier for A if and only if the vector field mA is without divergence i.e. $div\,(mA) = 0$. Thus, the set of last multipliers is a "measure of how far away" is A from being divergence-free.

(vi) To the vector field A one may associate an *adjoint vector field A^**, acting on functions in the following manner, [21, p. 129]:

$$A^*\,(m) = -A\,(m) - m\,divA.$$

Then, another simple characterization is: m is a last multiplier for A iff m is a first integral of the adjoint A^*. An important consequence results: the set of last multipliers is a subalgebra in $C^\infty(M)$.

An important structure generated by a last multiplier is given by:

Proposition 1.2. *Let $m \in C^\infty(M)$ be fixed. The set of vector fields admitting m as last multiplier is a Lie subalgebra in $\mathcal{X}(M)$.*

PROOF. Let X and Y be vector fields with the required property. Since [12, p. 123]:

$$div\,[X,Y] = X\,(divY) - Y\,(divX)$$

one has:

$$[X,Y]\,(m) + m\,div\,[X,Y] = (X\,(Y\,(m)) + m\,X\,(divY)) - (Y\,(X\,(m)) + m\,Y\,(divX)) =$$

$$= (-divY \cdot X\,(m)) - (-divX \cdot Y\,(m)) = divY \cdot mdivX - divX \cdot mdivY = 0.$$

$\square$

2. Last multipliers on Poisson manifolds

Let us assume that M is endowed with a Poisson bracket $\{,\}$. Let $f \in C^\infty\,(M)$ and $A_f \in \mathcal{X}\,(M)$ be the associated *Hamiltonian vector field* of the *Hamiltonian* f cf. [12]. Recall that given the volume form V there exists a unique vector field X_V, called the *modular vector field*, so that ([9], [24]):

$$div_V A_f = X_V\,(f)\,.$$

The triple $(M, \{,\}, V)$ is called ([24]) *unimodular* if X_V is a Hamiltonian vector field, A_ρ of $\rho \in C^\infty\,(M)$.

From (1.2) there results:

$$0 = A_f\,(m) + mX_V\,(f) = -A_m\,(f) + mX_V\,(f)$$

which means:

Proposition 2.1. *m is a last multiplier of A_f if and only if f is a first integral for the vector field $mX_V - A_m$. In the unimodular case, m is a last multiplier for A_f if and only if $m\{\rho, f\} = \{m, f\}$.*

Since f is a first integral of A_f we get:

Corollary 2.2. *f is a last multiplier for A_f if and only if f is a first integral of the vector field X_V. In the unimodular case, f is a last multiplier for A_f if and only if $\{\rho, f\} = 0$.*

With the Jacobi and Leibnitz formulas the following consequence of Corollary 2.2. may be established:

Corollary 2.3. *Let $(M, \{,\})$ be a unimodular Poisson manifold and let F be the set of smooth functions f that are last multipliers of A_f. Then F is a Poisson subalgebra in $(C^\infty\,(M), \cdot, \{,\})$.*

Another important consequence of Proposition 2.1. is:

Corollary 2.4. *If m is a last multiplier of A_f and A_g then m is a last multiplier of A_{fg}. Then, if m is a last multiplier of A_f then m is a last multiplier of A_{f^r} for every natural number $r \geq 1$.*

Let $(x^1, \ldots, x^n)$ be a local chart on M such that $V = dx^1 \wedge \ldots \wedge dx^n$ and the bivector π of $(M, \{,\})$ is: $\pi = \sum\limits_{i<j} \pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$. Denoting $\pi^i = \sum\limits_{j=1}^{n} \frac{\partial \pi^{ij}}{\partial x^j}$ we have ([5, Proposition 1, p. 4]):

$$X_V = \sum_{i=1}^{n} \pi^i \frac{\partial}{\partial x^i} \tag{2.1}$$

and then, Proposition 2.1. and Corollary 2.2. become:

Proposition 2.5. $m \in C^\infty (M)$ *is a last multiplier for A_f if and only if:*

$$m \sum_{i=1}^{n} \pi^i \frac{\partial f}{\partial x^i} = \{m, f\} = \sum_{i<j} \pi^{ij} \frac{\partial m}{\partial x^i} \frac{\partial f}{\partial x^j}. \tag{2.2}$$

$f \in C^\infty (M)$ *is a last multiplier for A_f if and only if:*

$$\sum_{i=1}^{n} \pi^i \frac{\partial f}{\partial x^i} = 0. \tag{2.3}$$

Examples:

2.1.

After [23, p. 31] the bivector $\pi = h(x, y) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ defines a Poisson structure on $\mathbb{R}^2$. So, $\pi^1 = \frac{\partial h}{\partial y}, \pi^2 = -\frac{\partial h}{\partial x}$ and then (2.3) becomes:

$$\frac{\partial h}{\partial y} \frac{\partial f}{\partial x} - \frac{\partial h}{\partial x} \frac{\partial f}{\partial y} = 0$$

with the obvious solution $f = h$. Therefore, on the Poisson manifold $(\mathbb{R}^2, \pi)$ above the function h is a last multiplier for exactly the Hamiltonian vector field A_h.

2.2. Lie-Poisson structures

Let $\mathcal{G}$ be an n-dimensional Lie algebra with a fixed basis $B = \{e_i\}_{1 \leq i \leq n}$ and let $B^* = \{e^i\}$ be the dual basis on the dual $\mathcal{G}^*$. Recall the definition of *structure constants* of $\mathcal{G}$:

$$[e_i, e_j] = c_{ij}^k e_k.$$

Then, on $\mathcal{G}^*$ we have the so-called *Lie-Poisson structure* given by ([23, p. 31]):

$$\pi^{ij} (x_u e^u) = c_{ij}^k x_k. \tag{2.4}$$

Hence:

$$\pi^i = \sum_{j=1}^{n} c_{ij}^j \tag{2.5}$$

yields:

Proposition 2.6. *For a Lie-Poisson structure on $\mathcal{G}^*$ provided by the structural constants $\left(c_{ij}^k\right)$ a function $f \in C^\infty\left(\mathcal{G}^*\right)$ is a last multiplier for A_f if and only if:*

$$\sum_{i,j=1}^n c_{ij}^j \frac{\partial f}{\partial x_i} = 0. \tag{2.6}$$

Particular case: n=2

From $[e_1, e_1] = [e_2, e_2] = 0, [e_1, e_2] = c_{12}^1 e_1 + c_{12}^2 e_2$ the last equation reads:

$$c_{12}^2 \frac{\partial f}{\partial x_1} - c_{12}^1 \frac{\partial f}{\partial x_2} = 0. \tag{2.7}$$

For example, if $c_{12}^1 \cdot c_{12}^2 \neq 0$, a solution of (2.7) is:

$$f = f(x_1, x_2) = A\left(\frac{x_1}{c_{12}^1} + \frac{x_2}{c_{12}^2}\right) + B \tag{2.8}$$

with A, B real constants.

3. The Riemannian case

Let us suppose that a Riemannian metric $g = <,>$ on M is given; then, there exists an induced volume form V_g. Let $\omega \in \Lambda^1(M)$ be the g-dual of A and δ the co-derivative operator $\delta : \Lambda^*(M) \to \Lambda^{*-1}(M)$. Then:

$$\begin{cases} div_{V_g} A = -\delta\omega \\ A(m) = g^{-1}(dm, \omega) \end{cases}$$

and condition (1.2) means:

$$g^{-1}(dm, \omega) = m\delta\omega.$$

Supposing that $m > 0$ it follows that m is a last multiplier if and only if ω belongs to the kernel of the differential operator: $g^{-1}(d\ln m, \cdot) - \delta : \Lambda^1(M) \to \Lambda^0 = C^\infty(M)$.

Now, assume that the vector field A admits a Helmholtz type decomposition:

$$A = X + \nabla u \tag{3.1}$$

where X is a divergence-free vector field and $u \in C^\infty(M)$; for example, if M is compact such decompositions always exist. From $div_{V_g} \nabla u = \Delta u$, the Laplacian of u, and $\nabla u(m) = <\nabla u, \nabla m>$ there follows that (1.2) becomes:

$$X(u) + <\nabla u, \nabla m> + m \cdot \Delta u = 0. \tag{3.2}$$

Example 3.1.

u is a last multiplier of $A = X + \nabla u$ if and only if:

$$X(u) = -u \cdot \Delta u - <\nabla u, \nabla u>.$$

Suppose that M is a cylinder $M = I \times N$ with $I \subseteq \mathbb{R}$ and N a $(n-1)$-manifold; then, for $X = -\frac{1}{2}\frac{\partial}{\partial t} \in \mathcal{X}(I)$ which is divergence-free with respect to $V = dt \wedge V_N$ with V_N a volume form on N, the previous relation yields:

$$u_t = 2\left(u \cdot \Delta u + <\nabla u, \nabla u>\right).$$

By the well-known formula ([18, p. 55]):

$$<\nabla f, \nabla g> = \frac{1}{2}\left(\Delta(fg) - f \cdot \Delta g - g \cdot \Delta f\right) \tag{3.3}$$

the previous equation becomes:

$$u_t = \Delta\left(u^2\right) \tag{3.4}$$

which is a nonlinear parabolic equation of the type of porous medium equation ([1]).

Example 3.2.

Returning to (3.1), suppose that $X = 0$. The condition (3.2) reads:

$$m \cdot \Delta u + <\nabla u, \nabla m> = 0 \tag{3.5}$$

which is equivalent, via (3.3) to:

$$\Delta(um) + m \cdot \Delta u = u \cdot \Delta m. \tag{3.6}$$

Adding to (3.6) a similar relation with u replaced by m leads to the following conclusion:

Proposition 3.1. *Let $u, m \in C^\infty(M)$ such that u is a last multiplier of ∇m and m is a last multiplier of ∇u. Then $u \cdot m$ is a harmonic function on M. $u \in C^\infty(M)$ is a last multiplier of $A = \nabla u$ if and only if u^2 is a harmonic function on M.*

If M is an orientable compact manifold then, from Proposition 3.1., it follows that u^2 is a constant which implies that u is a constant, too. But then $A = \nabla u = 0$. Therefore, on an orientable compact manifold, a function cannot be a last multiplier of its gradient vector field.

If $(M, g) = (\mathbb{R}^n, can)$, there are two classes (with respect to the sign $\pm$) of radial functions with harmonic square:

i) $n = 2$

$$u_\pm(r) = \pm\sqrt{C_1 lnr + C_2}$$

ii) $n = 1$ and $n \geq 3$

$$u_\pm(r) = \pm\sqrt{C_1 r^{2-n} + C_2}$$

with C_1, C_2 real constants and $r = \sqrt{(x^1)^2 + \ldots + (x^n)^2}$.

Example 3.3. The gradient of the distance function with respect to a two dimensional rotationally symmetric metric

Let M be a two dimensional manifold with local coordinates (t, θ) endowed with a *rotationally symmetric* metric $g = dt^2 + \varphi^2(t)d\theta^2$ cf. [18, p. 11]. Let $u \in C^\infty(M)$, $u(t, \theta) = t$, which appears as a distance function with respect to the given metric. Then, $\nabla u = \frac{\partial}{\partial t}$ and $\Delta u = \frac{\varphi'(t)}{\varphi(t)}$; equation (3.5) is:

$$m \cdot \frac{\varphi'(t)}{\varphi(t)} + \frac{\partial m}{\partial t} = 0$$

with solution: $m = m(t) = \varphi^{-1}(t)$.

This latter function has a geometric significance: let $T = T(t)$ be an integral of m i.e. $\frac{dT}{dt} = m = \frac{1}{\varphi(t)}$. Then, in the new coordinates (T, θ) the given metric is conformally Euclidean: $g = \varphi^2(t)\left(dT^2 + d\theta^2\right)$ where $t = t(T)$.

Example 3.4.

Another way to treat the case $X = 0$ is via equation (1.4):

$$div(m\nabla u) = 0 \tag{3.7}$$

in which the transformation $v = u\sqrt{m}$ (recall that we search for $m > 0$) is considered. From $div(m\nabla \frac{v}{\sqrt{m}}) = 0$, it results $div(\sqrt{m}\nabla v - v\nabla\sqrt{m}) = 0$ i.e.:

$$\sqrt{m}\Delta v = v\Delta\sqrt{m} \tag{3.8}$$

which yields:

Proposition 3.2. *Let $a > 0, b$ be solutions of Helmholtz (particularly Laplace) on the Riemannian manifold (M, g). Then, $m = a^2$ is a last multiplier for the gradient vector field of function $u = \frac{b}{a}$.*

4. Applications to gas dynamics

Consider again the Riemannian manifold (M, g). Set $m \in C^\infty(M)$ and recall according to [20, p. 62]:

Definition 2. A form ω is said to be:
(i) *m-coclosed* if $\delta(m\omega) = 0$,
(ii) *m-harmonic* if it is closed and m-coclosed.

An important result from the cited paper is:

Proposition 4.1. *If the 1-form ω is m-harmonic, then, locally, $\omega = d\phi$, where $\phi \in C^\infty(M)$ satisfies:*

$$\delta(md\phi) = 0. \tag{4.1}$$

In local coordinates $(x^1, ..., x^n)$ on M this equation has the form:

$$\frac{1}{\sqrt{detg}} \frac{\partial}{\partial x^i} (\sqrt{detg} g^{ij} m \frac{\partial \phi}{\partial x^j}) = 0$$

and for a flat M the last equation is exactly the classical gas dynamics equation conform [20, p. 63].

But (4.1) is exactly $div_{V_g}(m\nabla\phi) = 0$, which means that m is a last multiplier for the gradient vector field $\nabla\phi$. So, the last result can be rephrased:

Proposition 4.2. *If the 1-form ω is m-harmonic, then, locally, $\omega = d\phi$, where $\phi \in C^\infty(M)$ with $\nabla\phi$ having m as a last multiplier.*

Due to the local character of previous result for this setting, the Proposition 1.2. can be improved:

Proposition 4.3. *Let the 1-form ω be m-harmonic and $a, b \in C^\infty(M)$, such that $\omega = da = db$. Then m is a last multiplier for ∇a and ∇b but first integral for the vector field $[\nabla a, \nabla b]$.*

PROOF. We have:

$$[\nabla a, \nabla b] (m) = \nabla a (m\delta\omega) - \nabla b (m\delta\omega) = m [\nabla a (\delta\omega) - \nabla b (\delta\omega)].$$

But $\nabla a (\delta\omega) = \nabla b (\delta\omega) = g^{-1} (d\delta\omega, \omega)$. $\square$

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